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Youla-Kučera Parameterization: Theory and Applications

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Outline

Youla-Kučera parameterization refers to parameterizing all feedback controllers that can *stabilize* a given plant.

The presentation will cover

- **motivation, basic theory, historical notes**
- **simple applications to optimal control and multitask control**
- **advanced applications to robust stabilization and response shaping**
- **low-order stabilizing controllers**
- **transfer-matrix parametrization formula**
- **state-space realization of all stabilizing controllers**
- **industrial applications**

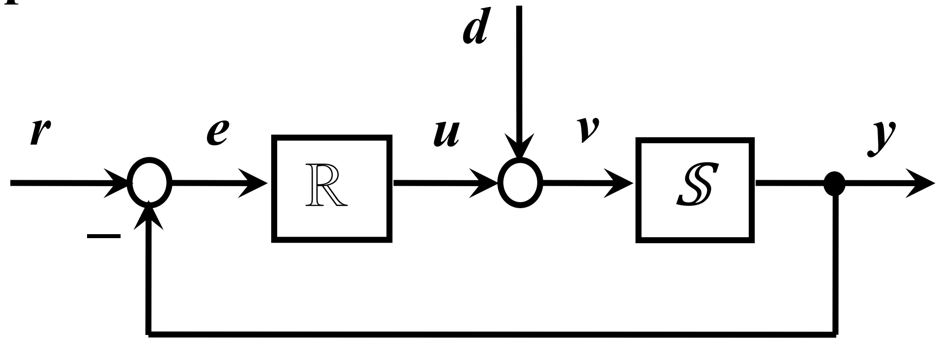
A typical control problem

Given a plant $\mathbb{S}$, determine a feedback controller $\mathbb{R}$ so that

(1) the closed-loop system is stable, and

(2) meets additional performance specifications.

The generic feedback system:



Stabilizing the system first

and then addressing the additional specifications one at a time is logical.

To do this, all solutions must be determined before proceeding to the next step;

this is the reason why we need to have all stabilizing controllers available.

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Systems and stability

We consider linear, time-invariant, differential *systems* $\mathbb{S}$ of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t), \quad t \geq 0$$

where u, x, and y are the input, state, and output vectors,

and A, B, C, and D are real matrices of appropriate sizes.

A system $\mathbb{S}$ gives rise to the *transfer function*

$$S(s) = C(sI - A)^{-1}B + D.$$

which is a *proper rational* matrix.

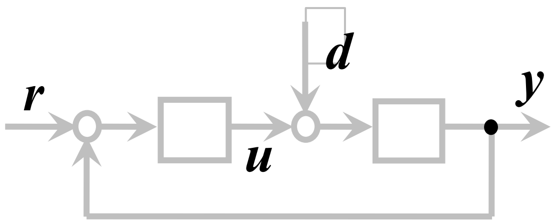
A system is considered *stable* if any initial state $x(0)$ goes to zero as $t \to \infty$.

A system is stable if and only if all eigenvalues of A have a negative real part.

A *controllable* and *observable* system is stable

if and only if all the poles of the transfer function have a negative real part.

Feedback system



The feedback system with inputs *d, r* and outputs *y, u* is *controllable* and *observable* whenever the constituent systems $\mathbb{S}$ and $\mathbb{R}$ are so.

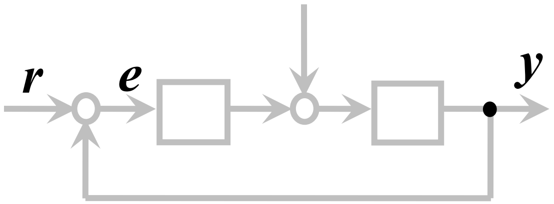
The transfer function, *H*, that relates *d, r* and *y, u* is assumed to be well defined; it is given by

$$H = \begin{bmatrix} S & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I & R \\ -S & I \end{bmatrix}^{-1} = \begin{bmatrix} R & I \\ I & -S \end{bmatrix}^{-1} \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}$$

where *S* and *R* are the transfer functions of the systems $\mathbb{S}$ and $\mathbb{R}$.

The controllable and observable feedback system is stable if and only if *H* is a *proper and stable* rational matrix (all poles within the open left half-plane).

Single-input, single-output systems



Write $S = BA^{-1}$ and $R = QP^{-1}$
as coprime, *proper and stable* fractions.
Coprime means having no unstable and infinite common zeros.

Define sensitivity function $H_S : r \rightarrow e$

$$H_S = \frac{1}{1+SR} = A\frac{P}{AP+BQ} := AX$$

and complementary sensitivity $H_C : r \rightarrow y$

$$H_C = \frac{SR}{1+SR} = B\frac{Q}{AP+BQ} := BY$$

In a *stable* closed-loop system, X and Y are *proper and stable* rational functions.
However, X and Y cannot be arbitrary since $H_S + H_C = 1$.
Therefore,

$$AX + BY = 1.$$

Youla-Kučera parameterization

Let $S = BA^{-1}$ be a coprime, proper and stable rational fraction for the plant.

Let X, Y be a *proper and stable rational* solution pair of the *Bézout equation*

$$AX + BY = 1.$$

Then, all controllers that stabilize the closed-loop system are given by

$$R = (X + BW)^{-1}(Y - AW)$$

where W is a *proper and stable rational* parameter

such that $(X + BW)^{-1}$ exists and is proper (so that R is proper).

Indeed, define $P := X + BW$, $Q := Y - AW$ so that $R = P^{-1}Q$.

Then, the closed-loop system transfer function H has the denominator

$$AP + BQ = AX + BY + (AB - BA)W = 1.$$

Hence, H is proper and stable rational.

Example 1

Consider a pure integrator plant with the transfer function $S = 1/s$.

In terms of proper and stable rational fractions, we have $S = BA^{-1}$, where

$$A = \frac{s}{s+\alpha}, \quad B = \frac{1}{s+\alpha},$$

for an arbitrary but fixed real number $\alpha > 0$.

The Bézout equation $AX + BY = 1$ admits the solution $X = 1$, $Y = \alpha$,

and the set of stabilizing controllers having a *proper rational* transfer function is given by

$$R = \left(1 + \frac{1}{s+\alpha} W\right)^{-1} \left(\alpha - \frac{s}{s+\alpha} W\right)$$

for any proper and stable rational W.

Technical notes

The Youla-Kučera parameterization is a fundamental result of control theory.

There is a *one-to-one correspondence*
between the set of parameters *W* and the set of stabilizing controllers *R*.

For any given plant *S*, the set of stabilizing controllers is infinite,
of the same cardinality as the set of proper and stable rational functions.

For any given plant *S*, finding stabilizing controllers
of *arbitrarily high order* is possible.

The most important bonus is that all the transfer functions
of a stable closed-loop system are *affine* in *W* while they are *nonlinear* in *R*,
which makes it easier to determine controllers through the parameter.

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Historical notes

D.C. Youla (1925-2021) from the Polytechnic Institute of New York University and V. Kučera discovered the parameterization *independently* in the mid-1970s. V. Kučera published the parameterization formula, while D.C. Youla showed how to utilize the parameter in the design of quadratic optimal controllers.

M. Vidyasagar provided a comprehensive account of the result ten years later. B.D.O. Anderson coined the term "*Youla-Kučera parameterization*" in his plenary lecture "A homage to Youla and Kucera" at the 1996 IFAC Congress. A. Quadrat generalized the results to a class of infinite-dimensional systems. I. Mahtout et al. collected the latest developments and industrial applications.

The Youla-Kučera parameterization has a dedicated Wikipedia article, and thousands of Google results are related to it.

Dual parameterization

The role of the two systems, $\mathbb{S}$ and $\mathbb{R}$, can be reversed.

So, there is a dual parameterization,

which describes all linear systems stabilized by a given linear controller.

The parameter can then describe plant variations.

This is useful for solving the problem of closed-loop plant identification.

Open-loop identification is more straightforward,

but it is often prohibitive to disconnect the plant.

Identifying the dual parameter instead of the plant itself

is then a *linear* problem like open-loop identification, see Hansen et al. (1989).

We shall focus on the original Youla-Kučera parameterization.

H_2 optimal control

Plant $S = BA^{-1}$, where A, B are coprime, proper and stable rational functions.

The task is to find a stabilizing controller $R = (X + BW)^{-1}(Y - AW)$,

such that a designated closed-loop transfer function,

say, $H_C = SR(1 + SR)^{-1} = B(Y - AW)$

has the least H_2 norm, defined by $\|H_C\|_2^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty}|H_C(i\omega)|^2\, d\omega.$

We suppose that both A and B have no zeros on the imaginary axis.

The norm is minimized using *inner-outer factorization* of $U := AB$,

then stable-unstable partial fraction decomposition of $V := U_i^{-1}BY$,

and completing the squares to obtain the unique optimal parameter $W = U_i^{-1}V_s$.

The consequent minimum value of the norm equals $\|H_C\|_2 = \|V_u\|_2$.

H_∞ optimal control

Plant $S = BA^{-1}$, where A, B are coprime, proper and stable rational functions.

The task is to find a stabilizing controller

such that a designated closed-loop transfer function,

say, $H_C = SR(1 + SR)^{-1} = B(Y - AW)$

has the least H_∞ norm, defined by $\left\| H_C \right\|_\infty := \sup_\omega \left| H_C(i\omega \right|$.

We suppose that both A and B have no zeros on the imaginary axis.

If, in addition, AB has only one unstable zero, say at $s = s_0$,

then the unique optimal parameter is

$$W = \frac{BY - BY(s_0)}{AB}$$

and $\left\| H_C \right\|_\infty = \left| BY(s_0) \right|$.

In general, the optimal parameter W is obtained

by solving a *Nevanlinna-Pick interpolation* problem.

Asymptotic properties

Reference tracking:

The output *y* follows a reference signal *r* (error *e* goes to zero) asymptotically.

Expressed in terms of the Laplace transform, $L(e) := \bar{e}$**,**

$\bar{e} = H_S \bar{r}$ **is to be a proper and stable rational function.**

Disturbance attenuation:

The effect of a disturbance *d* on the output *y* decreases asymptotically.

Expressed in terms of the Laplace transform,

$\bar{y} = S H_S \bar{d}$ **is to be a proper and stable rational function.**

This is to be achieved by selecting a parameter *W*.

Example 2

Plant $S = (s + 1)/s = BA^{-1}$, where $A = s/(s + 1)$ and $B = 1$.

The set of stabilizing controllers is

$$R = \left(1 - \frac{s}{s+1} W\right) W^{-1}$$

for any proper and stable rational W such that W^{-1} exists and is proper.

The achievable sensitivity transfer functions are $H_S = \frac{s}{s+1} W$.

To track step references, $\bar{r} = k / s$, k real number, we have $\bar{e} == Wk / (s + 1)$,

which imposes *no further constraint* on W.

To attenuate sinusoidal disturbances, $\bar{d} = (as + b) / (s^2 + \omega^2)$, a, b real numbers,

we must *further constrain* the parameter as $W = W_1(s^2 + \omega^2)/(s + 1)^2$

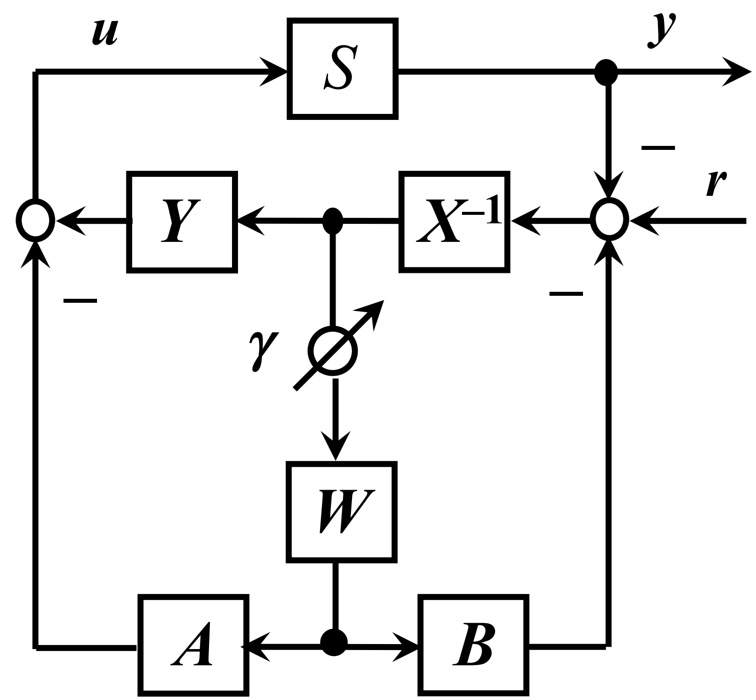
for any proper and stable rational W_1 such that W_1^{-1} is proper.

This demonstrates the *internal model principle*.

Multitask control

The merit of the Youla-Kučera parameterization is the possibility of switching among several controllers to meet different, often conflicting, requirements while retaining stability.

The Youla-Kučera controller structure is as follows,



where the scalar factor $\gamma \in [0,1]$ facilitates a bumpless switching.

Industrial application – autonomous vehicle

Mahtout et al. (2018) presented a *lateral controller for an autonomous vehicle*.

The vehicle lateral dynamics model has

- **four states: lateral position, lateral velocity, yaw angle, and yaw rate;**
- **one output: lateral position;**
- **one input: steering angle in the front tire**

plus steering actuator dynamics.

Two controllers are designed based on the target point method:

- **lane changing controller $\mathbb{R}_1$ – must be smooth to avoid overshoots and uncomfortable sensation in the vehicle, the target point is set to 30m;**
- **lane tracking controller $\mathbb{R}_2$ – must be fast, the closer is the target point, the smaller is the tracking error, the look-ahead distance is fixed to 15m.**

Industrial application – autonomous vehicle

The parameter γ is in charge of the controller's switching based on the vehicle lateral error with respect to the trajectory as follows:

- **When the lateral error > 3m , the adequate controller is $\mathbb{R}_1$ so $\gamma = 0$.**
- **When the lateral error < 0.2m , the adequate controller is $\mathbb{R}_2$ so $\gamma = 1$.**
- **Between the two limits γ changes gradually depending on the lateral error.**

The proposed approach was tested on an electric Renault Zoe that had been modified for allowing steering computed control. Experimental results have proven that the proposed control structure enhances the performance of only using a single controller for both cases. Errors are significantly reduced and the operation is smooth.

Robust stabilization

Robust stabilization is a technique that involves using a *fixed* controller to stabilize plants that are subject to modeling errors when the actual plant may differ from the nominal one.

The objective is to stabilize the actual plant.
Since the actual plant is unknown, however,
the best approach is to stabilize a large enough set of plants,
which is meticulously constructed as a neighborhood of the nominal plant.

The size of the neighborhood is measured by a suitable norm,
with the most common being the H_∞ norm.

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Model of uncertainty

Consider a nominal plant $\mathbb{S}$ with transfer function S

and its neighborhood $\mathbb{S}_\Delta$, defined by $S_\Delta := (1 + \Delta F)S$.

Here, F is a *fixed*, stable rational function,

and Δ is a *variable* stable rational function such that $\|\Delta\|_\infty \leq 1$.

Note that ΔF represents the normalized plant perturbation away from 1:

$$S_\Delta / S = 1 + \Delta F.$$

Then, for all frequencies ω, we have

$$\left| \frac{S_\Delta(i\omega)}{S(i\omega)} - 1 \right| \leq F(i\omega)$$

so $|F(j\omega)|$ provides the *uncertainty profile*,

while Δ accounting for *phase uncertainty*.

Robust stability condition

Suppose that a controller $\mathbb{R}$ stabilizes the nominal plant $\mathbb{S}$.

Then, by the Small Gain Theorem, $\mathbb{R}$ will stabilize the entire family of plants $\mathbb{S}_\Delta$ if and only if

$$\left\| \frac{SR}{1+SR} F \right\|_\infty < 1.$$

When the stabilizing controllers $\mathbb{R}$ are expressed in terms of the parameter W, the robust stability condition reads as follows:

$$\left\| B(Y - AW) \right\|_\infty < 1.$$

Any proper and stable rational W that satisfies this inequality defines a controller that robustly stabilizes the nominal plant $\mathbb{S}$.

Example 3

Consider a nominal plant with the transfer function $S = (s + 1)/(s - 1)$

We know there is a delay ϑ in the system,

which falls within the interval $0 \le \vartheta \le 0.2$.

Therefore, we embed the system

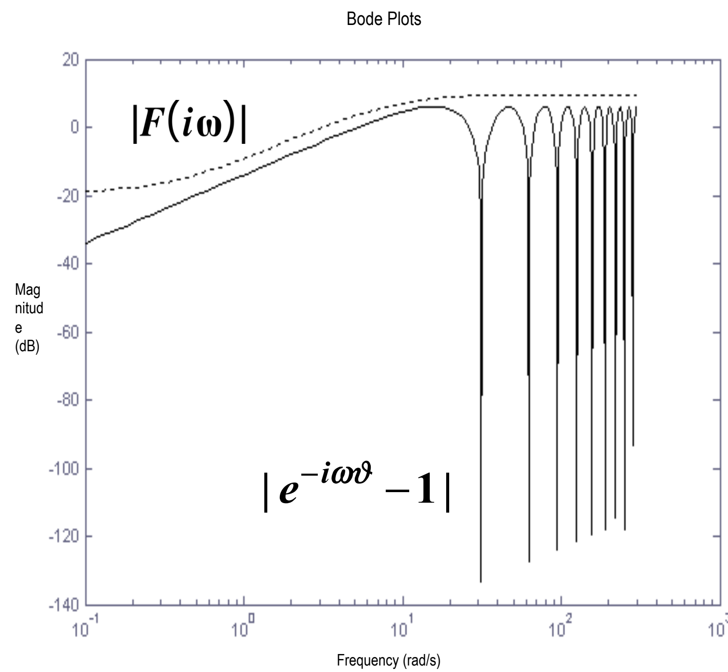
in the system class

$$S_\Delta := \left\{ \frac{s+1}{s-1} e^{-\vartheta s} \mid 0 \le \vartheta \le 0.2 \right\}.$$

The relative

plant uncertainty

$$\frac{S_\Delta}{S} - 1 = e^{-i\omega\vartheta} - 1$$



can be majorized in amplitude by the transfer function $F = (3s + 1)/(s + 9)$.

Example 3

The controllers responsible

for stabilizing the nominal plant,

$$S = \frac{s+1}{s-1} = \frac{B}{A}$$

with

$$A = \frac{s-1}{s+1}, \quad B = 1,$$

are determined by solving the Bézout equation $AX + BY = 1$.

Explicitly, $X = 0$, $Y = 1$ and

$$R = \frac{1 - \frac{s-1}{s+1} W}{W},$$

where W is a proper and stable rational parameter

such that W^{-1} exists and is proper.

Example 3

The robust stability condition is

$$\left\| B(Y - AW)F \right\|_\infty := \left\| N - MW \right\|_\infty < 1.$$

where

$$N = BYF = \frac{3s+1}{s+9}, \quad M = BAF = \frac{s-1}{s+1}\frac{3s+1}{s+9}.$$

We calculate $\min_W \left\| N - MW \right\|_\infty = N(1) = 0.4,$ **which is less than 1.**

The minimizing parameter

$$W = \frac{N - N(1)}{M} = 2.6\frac{s+1}{3s+1}$$

corresponds to the robust stabilizing controller

$$R = \frac{4}{26}\frac{s+9}{s+1}.$$

Since the norm is not only less than one but also minimal,

then $\mathbb{R}$ is considered the *best* robust stabilizing controller.

Pole placement

A stable system has poles anywhere in the stability region;
a selection of the parameter *W* can achieve specific locations.

The control system performance is specified by a *pole polynomial,*
which is the *characteristic polynomial*
whenever the closed-loop system is controllable and observable.

Given a plant $S = BA^{-1}$, we write $S = b/a$ as a coprime *polynomial* fraction.
Let x, y be a *polynomial* solution of the equation $ax + by = 1$.
Then, the set of *s*tabilizing controllers can equivalently be expressed as

$$R = (y - aW) \,/\, (x + bW)$$

where *W* is a *stable* rational parameter such that *R* is *proper*.

Pole placement

Let the desired pole locations be specified by a stable pole polynomial *d*.

Write $W = w/d$ for a polynomial *w*. Then,

$$R = (dy - aw)\ /\ (dx + bw) := q/p$$

and the pole polynomial is

$$ap + bq = (ax + by)d + (ab - ba)w = d.$$

Thus, the denominator of the parameter W determines the closed-loop poles.

The polynomial *d* specifies the locations of the closed-loop poles,

while the polynomial *w* represents the remaining degrees of freedom.

Selecting *W*, we can achieve *any* polynomial *d*

having a sufficiently high degree (at least 2 deg *a* – 1).

Example 4

Given a plant with transfer function $S = 1/(s - 1)$,

we seek to assign the pole polynomial $d = s^2 + 2s + 1$.

The stabilizing controllers are

$$R = \frac{1-(s-1)W}{W}, \quad W \neq 0 \text{ stable rational.}$$

Put $W = w/d$.

Then,

$$R = \frac{(s^2+2s+1)-(s-1)w}{w}.$$

The plant has order 1.

For the closed-loop system to have order 2, the controller must have order 1.

Therefore, we have $w = s + \omega$ for any real ω.

Stabilization with fixed order controllers

A weak point of the design based on the Youla-Kučera parameterization is that each performance specification beyond stability may increase the order of the controller.

The degree control in the parameter $W = w/d$ is difficult. Fixed-order stabilizing controllers (presumably of low order) can be found by solving a *linear matrix inequality*.

Suppose a plant $S = b/a$ is given in terms of a polynomial fraction and suppose that we have a stabilizing controller $R = q/p$. We seek to find a stabilizing controller $R = y/x$ of a given order m whenever such a controller exists.

Minimal polynomial basis

The two stabilizing controllers are related as

$$p = x + bW, \quad q = y - aW, \quad \text{where } W = w/d.$$

Then all stabilizing controllers can be determined from the the minimal polynomial basis

$$\begin{bmatrix} d & 0 & -p & b \\ 0 & d & -q & -a \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ d_1 & d_2 \\ w_1 & w_2 \end{bmatrix} = 0$$

as

$$R = (\lambda_1 y_1 + \lambda_2 y_2) / (\lambda_1 x_1 + \lambda_2 x_2)$$

where λ_1 **and** λ_2 **are polynomials such that** $d := \lambda_1 d_1 + \lambda_2 d_2$ **is a stable polynomial.**

A stabilizing controller of order m **exists if**

$$\deg \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = m.$$

Convex inner approximation

Alas, the set of *stable* polynomials is not in general convex.
Given a fixed stable "central" polynomial $c(s)$ of degree n,
the polynomial $d(s)$ of degree n is stable
if a certain linear matrix inequality is satisfied.

The solution set of this inequality is a *convex inner approximation*
of the stability domain in the space of polynomial coefficients
around the central stable polynomial.

Optimizing over polynomials λ_1 and λ_2
we can enforce low degrees of x and y (linear algebraic constraint)
as well as the stability of d (linear matrix inequality constraint).

Example 5

Consider a plant of order 3,

$$S = \frac{1}{s(s^2 + s + 10)}.$$

A stabilizing controller of order 2 can be found by placing the closed-loop poles at arbitrary locations. For example, the controller

$$R = \frac{-26s^2 + 45s + 1}{s^2 + 4s - 4}$$

places all five closed-loop poles at -1.

We seek to find a lower-order stabilizing controller.

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Example 5

A minimal polynomial basis for the polynomial matrix relating the given and the target controllers is

$$\begin{bmatrix} 0 & 1 \\ -1 & -26 \\ -1 & s^3+s^2+10s-26 \\ s^2+4s-4 & 149s-103 \end{bmatrix}$$

All the stabilizing controllers can be recovered from the polynomials λ_1 and λ_2 such that the pole polynomial

$$d = -\lambda_1 + \lambda_2(s^3 + s^2 + 10s - 26)$$

is stable.

Example 5

From the first two rows of the basis a controller of order 0 can be obtained by restricting the parameters λ_1 and λ_2 to be constant.

$$\begin{bmatrix} 0 & 1 \\ -1 & -26 \\ -1 & s^3+s^2+10s-26 \\ s^2+4s-4 & 149s-103 \end{bmatrix}$$

Hurwitz stability criterion then reveals that d is stable if and only if $\lambda_1 \in (-36, -26)$ and $\lambda_2 = 1$.

For example, with $\lambda_1 = -30$ we obtain the controller $R = 4$ and the closed-loop pole polynomial $d = s^3 + s^2 + 10s + 4$.

In this example, we were able to obtain an exact solution.
In general, the linear matrix inequality provides a conservative solution.

Input and output shaping

Given a plant $S = b/a$, we seek a stabilizing controller $R = q/p$ such that the output y asymptotically follows a reference r while the *time-domain constraints* $u_{\min} \le u(t) \le u_{\max}$ and $y_{\min} \le y(t) \le y_{\max}$ are satisfied for all $t \ge 0$.

Can handle input constraints and also output overshooting or undershooting.

The approach is to assign negative integer poles multiples of, say σ, and express time signals as polynomials in the exponential modes $\lambda := \exp(-\sigma t)$. When time t increases from 0 to ∞, indeterminate λ decreases from 1 to 0, and the time constraints become a polynomial nonnegativity constraints. The satisfaction of these constraints is equivalent to solving a linear matrix inequality.

Example 6

Given the plant

$$S = \frac{s+0.5}{s(s-2)}.$$

The stabilizing controller

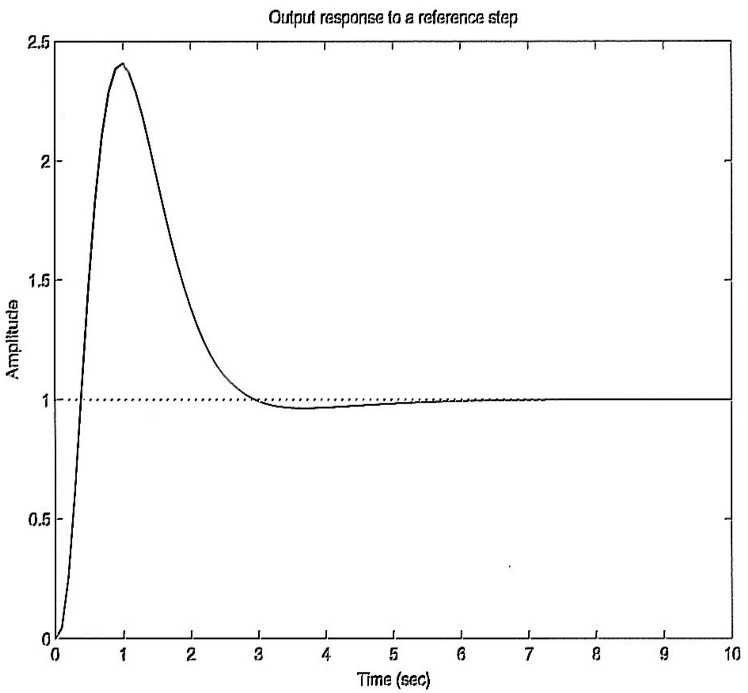
$$R = \frac{384s+240}{s^3+17s^2+119s+79}$$

assigns the closed-loop

poles at – 1, – 2, – 3, – 4, – 5

while ensuring asymptotic

step reference tracking.



Despite the poles being negative real,

the step response has an unacceptable overshoot of 140 % due to system zeros.

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Example 6

The set of all proper rational controllers that assign the above poles is given by

$$R = \frac{384s + 240 - s(s-2)w}{s^3 + 17s^2 + 119s + 79 + (s+0.5)w}$$

where $w = w_0 + w_1 s$ is a free polynomial of degree at most 1.

The closed-loop responses to a step input are affine in w,

$$\bar{y} = \frac{384s^2 + 423s + 120 - (s^3 - 1.5s^2 - s)w}{(s+1)(s+2)(s+3)(s+4)(s+5)}$$

and correspond to a sum of decaying exponential modes in the time domain.

The coefficients $\bar{y}$ are *linear* functions of w_0 and w_1.

Example 6

Suppose the desired maximum overshoot is 20%

$$y(t) \leq 1.2\, y_0$$

equivalent to the polynomial

non-negativity constraint

$$p(\lambda) = 1.2y_0 - y(\lambda) = 0.2y_0 - y_1\lambda - y_2\lambda^2 - y_3\lambda^3$$

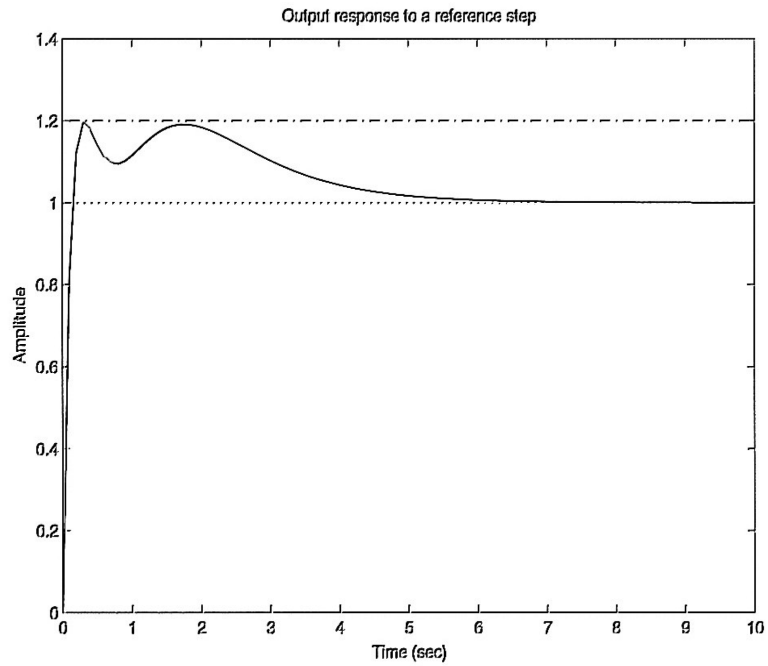
and in turn, equivalent

to a linear matrix inequality in w_0 and w_1.

The linear matrix inequality returns

$$w(s) = -\,100.36 - 12.27s$$

keeping the controller of order 3.



Introducing new control components

When new sensor or actuator hardware becomes available in a control system, it is possible to improve control performance through a redesign.
Rather than completely revamping the entire control system
and introducing new equipment,
it is often preferable to gradually replace the existing parts
while retaining the current control system in place.

The Youla-Kučera controller
allows for a smooth transition to the new controller
and provides the option to revert to the old controller if necessary.
This is a strong argument favoring the method from a practical perspective.

Industrial application – livestock stable climate control

Trangbaek and Bendtsen (2009) presented a *livestock-stable climate control,* for a stable located in Northern Jutland, Denmark.

A simple proportional-integral controller maintains a fixed temperature, but there is an undesirable air leakage into the stable.

The draft does not show on the temperature sensor but the livestock avoids that area.

Youla-Kučera controller is an elegant solution of how to integrate a new sensor, which detects the air leakage temperature into the existing control system while ensuring the stability of the transition.

Multi-input, multi-output systems

Now, the transfer functions S and R are proper rational *matrices*.

Write them in the form of *proper and stable rational matrix fractions*,

$$S = B_R A_R^{-1} = A_L^{-1} B_L$$

where A_R, B_R are *right coprime*, proper and stable rational matrices,

and A_L, B_L are *left coprime*, proper and stable rational matrices.

For example,

$$S = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s} \\ 0 & \frac{s+1}{s} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{s}{s+1} \end{bmatrix}^{-1} = B_R A_R^{-1}$$

$$= \begin{bmatrix} 1 & -\frac{1}{s+1} \\ 0 & \frac{s}{s+1} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & 1 \end{bmatrix} = A_L^{-1} B_L .$$

Parameterization of all stabilizing controllers

Let $$S = B_R A_R^{-1} = A_L^{-1} B_L$$

be right / left coprime, proper and stable rational matrix fractions.

Let X_R, Y_R and X_L, Y_L be proper and stable rational matrices such that the Bézout identity holds:

$$\begin{bmatrix} X_R & Y_R \\ -B_L & A_L \end{bmatrix} \begin{bmatrix} A_R & -Y_L \\ B_R & X_L \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Then, all controllers that stabilize the closed-loop system are given by

$$R = (X_R + WB_L)^{-1} (Y_R - WA_L) = (Y_L - A_R W) (X_L + B_R W)^{-1}$$

where W is a proper stable rational matrix parameter such that the indicated inverses exist and are proper.

State space representation of stabilizing controllers

Let

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

be a *controllable and observable* realization

of the system transfer function

$$S = C(sI - A)^{-1}B + D := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

This special notation helps to visualize

the connection between the transfer function and its state space realization.

We shall see that coprime, proper and stable matrix fractions for *S* and *R*
can be obtained *directly* from the matrices *A*, *B*, *C*, and *D*,
without solving the Bézout equations.

Right matrix fraction for S

Consider a stabilizing state feedback $u = Fx + \rho$

around the system

$$\dot{x} = (A + BF)x + B\rho$$
$$u = Fx + \rho$$
$$y = (C + DF)x + D\rho.$$

Define

$$A_R := \left[\begin{array}{c|c} A + BF & B \\ \hline F & I \end{array}\right], \quad B_R := \left[\begin{array}{c|c} A + BF & B \\ \hline C + DF & D \end{array}\right].$$

Then, in terms of the Laplace transform,

$$\bar{y} = B_R \bar{\rho}, \quad \bar{u} = A_R \bar{\rho}$$

so that

$$\bar{y} = B_R A_R^{-1} \bar{u} = S\bar{u}.$$

Left matrix fraction for S

Consider a state observer for the system $\dot{x} = Ax + Bu$, $y = Cx + Du$ based on a stabilizing output injection $K\varepsilon$

$$\dot{\xi} = (A - KC)\xi + (B - KD)u + Ky$$
$$\varepsilon = y - C\xi - Du.$$

Define

$$A_L := \left[\begin{array}{c|c} A - KC & K \\ \hline -C & I \end{array}\right], \quad B_L := \left[\begin{array}{c|c} A - KC & B - KD \\ \hline C & D \end{array}\right].$$

Then, in terms of the Laplace transform,

$$\bar{\varepsilon} = A_L \bar{y} - B_L \bar{u} = (A_L B_R - B_L A_R)\bar{\rho} = 0$$

so that

$$A_L^{-1} B_L = B_R A_R^{-1} = S.$$

Right matrix fraction for R

Consider a stabilizing state feedback $u = Fx \ (\rho = 0)$

around the observer with output y

$$\dot{\xi} = (A + BF)\xi + K\varepsilon$$
$$u = F\xi$$
$$y = (C + DF)\xi + \varepsilon.$$

Define

$$X_L := \left[\begin{array}{c|c} A+BF & K \\ \hline C+DF & I \end{array}\right], \quad Y_L := \left[\begin{array}{c|c} A+BF & K \\ \hline -F & 0 \end{array}\right].$$

Then, in terms of the Laplace transform,

$$\bar{u} = -Y_L\bar{\varepsilon}, \quad \bar{y} = X_L\bar{\varepsilon}$$

so that

$$\bar{u} = -Y_L X_L^{-1} = -R\bar{y}.$$

Left matrix fraction for *R*

Consider stabilizing state feedback $u = F\xi + \rho$

around the observer with output ρ

$$\dot{\xi} = (A - KC)\xi + (B - KD)u + Ky$$
$$\rho = -F\xi + u.$$

Define

$$X_R := \left[\begin{array}{c|c} A - KC & B - KD \\ \hline -F & I \end{array}\right], \quad Y_R := \left[\begin{array}{c|c} A - KC & K \\ \hline -F & 0 \end{array}\right].$$

Then, in terms of the Laplace transform,

$$\bar{\rho} = Y_R \bar{y} + X_R \bar{u} = (Y_R X_L - X_R Y_L)\bar{\varepsilon} = 0$$

and

$$X_R^{-1} Y_R = Y_L X_L^{-1} = R.$$

Coprime fractions

Collecting the equations,

$$\begin{bmatrix} \bar{r} \\ \bar{e} \end{bmatrix} = \begin{bmatrix} X_R & Y_R \\ -B_L & A_L \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{y} \end{bmatrix}, \quad \begin{bmatrix} \bar{u} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} A_R & -Y_L \\ B_R & X_L \end{bmatrix} \begin{bmatrix} \bar{r} \\ \bar{e} \end{bmatrix},$$

the Bézout identity follows

$$\begin{bmatrix} X_R & Y_R \\ -B_L & A_L \end{bmatrix} \begin{bmatrix} A_R & -Y_L \\ B_R & X_L \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Stabilizing controllers – transfer function

The coprime fractions determine an *observer-based controller* that stabilizes the system.

To determine *all* the stabilizing controllers, we introduce the parameter.

Put $\bar{\rho} = W\bar{\varepsilon}$.

Then

$$\begin{bmatrix} \bar{u} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} A_R & -Y_L \\ B_R & X_L \end{bmatrix} \begin{bmatrix} W\bar{\varepsilon} \\ \bar{\varepsilon} \end{bmatrix} = \begin{bmatrix} -(Y_L - A_R W)\bar{\varepsilon} \\ (X_L + B_R W)\bar{\varepsilon} \end{bmatrix}$$

and

$$0 = \begin{bmatrix} I & -W \end{bmatrix} \begin{bmatrix} \bar{r} \\ \bar{\varepsilon} \end{bmatrix} = (X_L + WB_L)\bar{u} + (Y_R - WA_L)\bar{y}.$$

Hence

$$R = (X_R + WB_L)^{-1}(Y_R - WA_L) = (Y_L - A_R W)(X_L + B_R W)^{-1}.$$

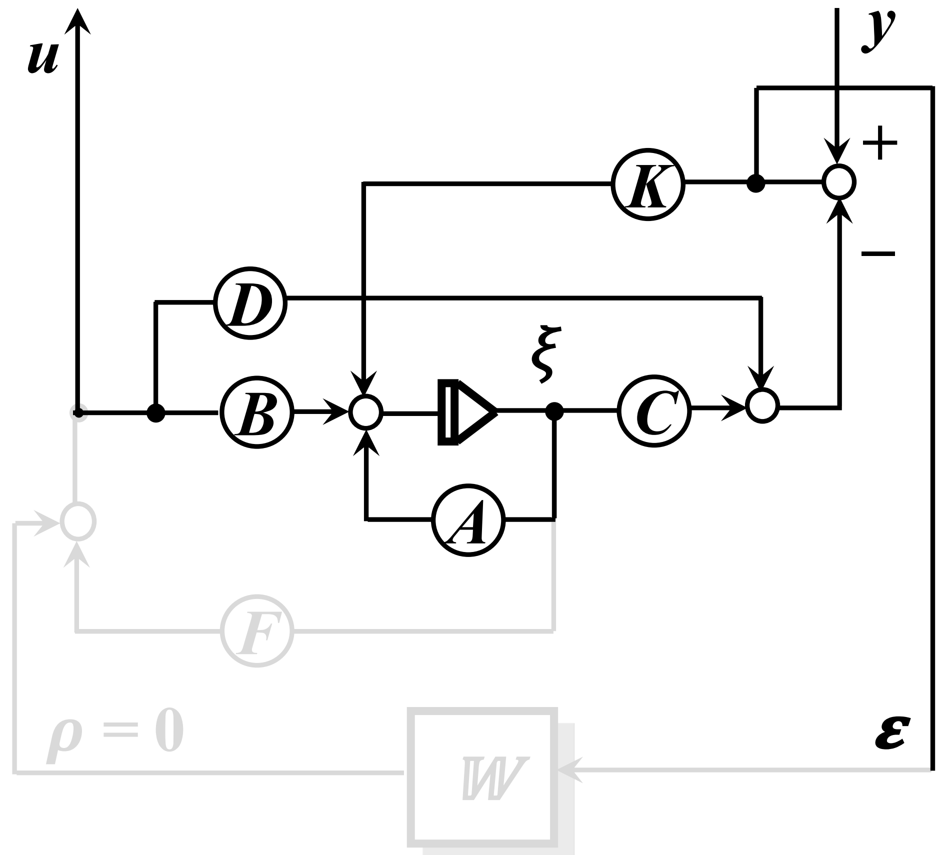
Stabilizing controllers – state space realization

All controllers that stabilize a given system are built around an observer-based controller:

observer

feedback

parameter



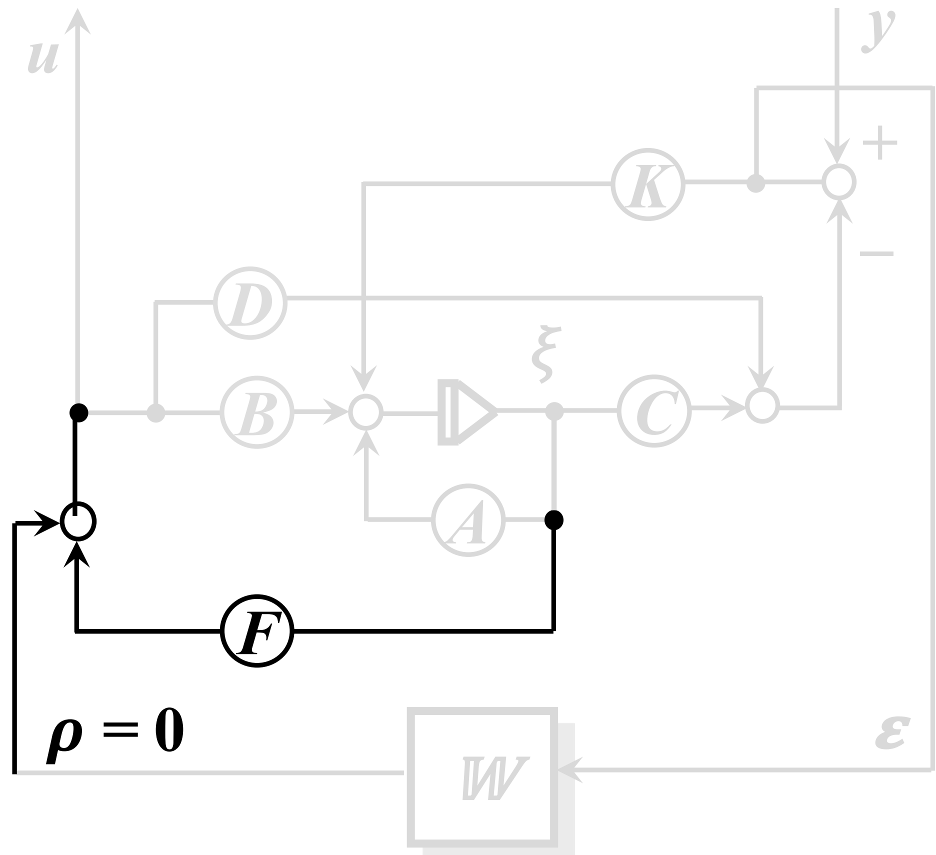
Stabilizing controllers – state space realization

All controllers that stabilize a given system are built around an observer-based controller:

observer

feedback

parameter



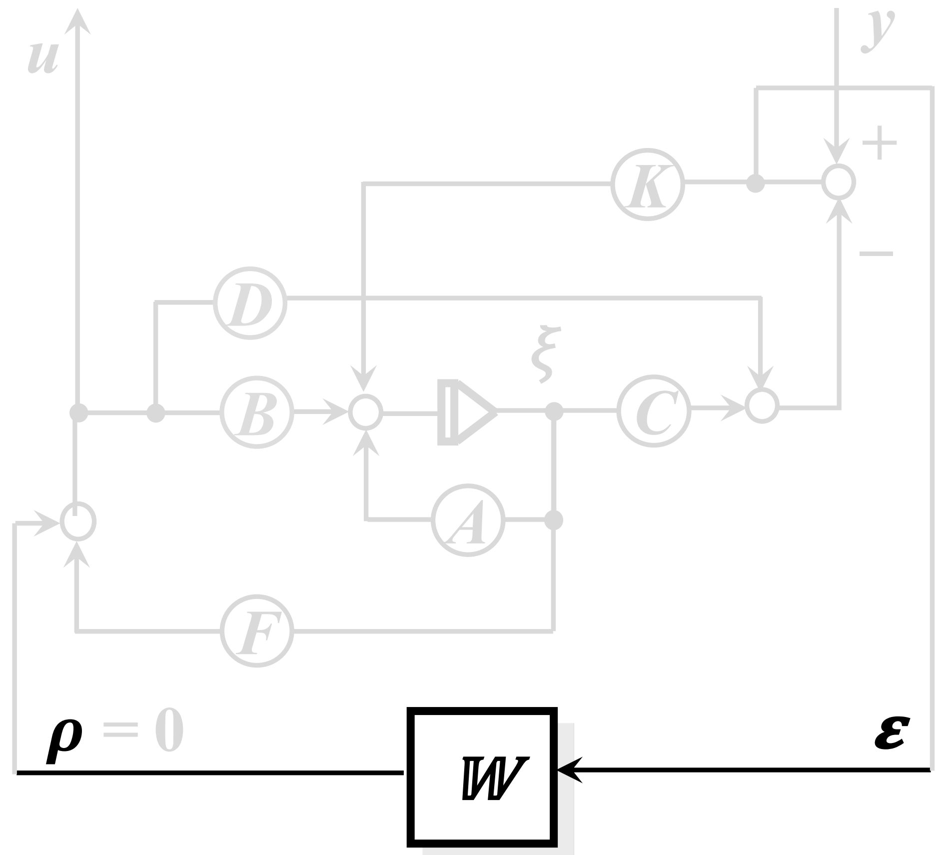
Stabilizing controllers – state space realization

All controllers that stabilize a given system are built around an observer-based controller:

observer

feedback

parameter



Stabilizing controllers – state space realization

All controllers that stabilize a given system are built around an observer-based stabilizing controller by adding the parameter. The parameter $\mathbb{W}$ is any stable system having the transfer function W.

There is no need to construct coprime fractions nor to solve Bézout equations. The observer-based controller is given directly by F and K. The controller's order equals the plant's order plus the order of $\mathbb{W}$.

Some stabilizing controllers may not be controllable or observable. When only the controllable and observable part is retained, the stabilizing controller has a lower order, but it has no longer a nice observer-based structure.

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Control of complex systems

Designing finite-dimensional, linear, time-invariant controllers
for infinite-dimensional, linear, time-invariant plants
can successfully be approached via Youla-Kučera parameterization.

The distributed parameter plant is first replaced
by a finite dimensional approximant.
The Youla-Kučera parameterization can then be used
to parameterize the set of all stabilizing linear, time-invariant controllers,
and a performance criterion is formulated.
The resulting finite-dimensional optimization problem,
possibly subject to time-domain constraints, is then used to obtain the solution.

Industrial application – irrigation system

Irrigation is an increasingly important issue worldwide.
Cifdaloz et al. (2008) use the Saint Venant equations
(nonlinear hyperbolic partial differential equations)
to describe the gravity-based fluid flow in canals and rivers.

The equations are linearized, resulting in time-delay transfer functions
relating the water level to be controlled and the control flow rates
at the upstream and downstream gates of the irrigation canal.
Padé approximants are then used to obtain
a finite-dimensional multivariable plant description.
The performance measure is a mixed-sensitivity H_∞ norm minimization
subject to constraints on the water discharge at the gates.

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